Comment on *Sixth-Order Vacuum-Polarization Contribution to the Lamb Shift of Muonic Hydrogen* by T. Kinoshita, and M. Nio, Phys. Rev. Lett. 82, 3240 (1999)

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Recently, while performing a calculation of the α^2 corrections to HFS interval in muonic hydrogen [1], we had to calculate contributions of the third order *bound-state* perturbation theory (PT). The general expression for those corrections is of the form (see, e.g., [2, 3])

$$\Delta E^{(3)}(ns) = \langle\Psi_{ns}|\delta V\widetilde{G}\Big[\delta V - \Delta E^{(1)}_{ns}\Big]\widetilde{G}\delta V|\Psi_{ns}\rangle\ , \quad (1)$$

where $\Delta E^{(1)}_{ns} = \langle\Psi_{ns}|\delta V|\Psi_{ns}\rangle$, δV is a sum of all perturbations under consideration and Ψ_{ns} and $\widetilde{G}$ are the wave function and the reduced Green function, respectively, of the unperturbed problem (i.e., of the non-relativistic Coulomb problem in our case).

In contrast to the scattering PT the bound-state PT contains certain subtractions. In particular, in the third order, the bound-state PT (see Eq. (1)) in addition to an ordinary contribution ($\sim\delta V$) involves one more term ($\sim\Delta E^{(1)}_{ns}$). We refer to the former as to a ‘main’ term and to the latter as to a ‘subtraction’ term.

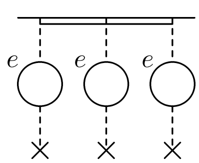

FIG. 1: The $\alpha^5 m$ correction to the Lamb shift in muonic hydrogen: the only contribution of the third order of non-relativistic perturbation theory (cf. Fig. 5c in [4])

For muonic hydrogen the α^2 contributions into the HFS interval are similar to one of the α^3 corrections to the Lamb shift (see Fig. 1), which was previously calculated for the $2s$ and $2p$ states in [4] (see the third line in Eq. (25) in [4]),

$$\Delta E(2p-2s) = 0.002535(1)\frac{\alpha^5}{\pi^3}m_r c^2\ , \quad (2)$$

where m_r is the reduced mass for muonic hydrogen and α stands for the fine structure constant.

To cross-check our calculations on HFS [1], we have also calculated contribution of diagram in Fig. 1 into the $2s-2p$ splitting and found

$$\begin{aligned}\Delta E(2p-2s) &= \Big[\big(-7.3861\cdot 10^{-6} + 0.3511\cdot 10^{-6}\big) \\ &\quad \left[-\big(-0.002\,5412 + 0.001\,3661\big)\right]\frac{\alpha^5}{\pi^3}m_r c^2 \\ &= 0.0011681\frac{\alpha^5}{\pi^3}m_r c^2\ , \quad (3)\end{aligned}$$

where the first parentheses are for the $2p$ contribution, while the second are for the $2s$ one; each consists of the main and the subtraction term as introduced in (1).

The results in (2) and (3) disagree. Our result confirms calculations [4] for the main terms for the $2s$ and $2p$ states, while the difference originates from the fact that the subtraction terms are missing in [4], as we later learned [5] from the authors of the paper.

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[2] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory: Vol. 3 (Course of Theoretical Physics)*. (Pergamon Press, Oxford, London, 1965) (see problem 2 in p. 132).
[3] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*. (Plenum Publishing, New York, 1977) (see Eq. (25.12) in p. 123).
[4] T. Kinoshita, and M. Nio, Phys. Rev. Lett. **82**, 3240, (1999)
[5] T. Kinoshita, and M. Nio, private communication